On a quantum group invariant spin chain with non-local boundary conditions

On a quantum group invariant spin chain with non-local boundary conditions

H Grosse†, S Pallua‡§, P Prester‡ and E Raschhofer†‖

† Institut für Theoretische Physik, Universität Wien, Boltzmanngasse 5, A-1090 Wien, Austria
‡ Department of Theoretical Physics, University of Zagreb POB 162, Bijenička cesta 32, 41000 Zagreb, Croatia

Abstract. We treat a one parameter family of quantum spin models, which are quantum-group invariant as well as periodic in a certain sense, using Bethe states. We analyse its ground-state properties and obtain the surprising result that its spin becomes non-zero depending on the value of the parameter. Finite-size corrections allow us to obtain the central charge which is also spin dependent. Relations to other models are mentioned.

1. Introduction

Integrable models are cornerstones for checking theoretical ideas. They are central in the many relationships between the topological Chern–Simons model, knot invariants, solutions of Yang–Baxter equations, WZNW models and aspects of conformal field theory. They led to the notion of quantum groups and to new ideas about symmetries.

Asking for quantum-group invariant one-dimensional quantum spin models, we learnt from [1] that very special boundary terms have to be considered. They break translational invariance. It is a natural question whether one can form a closed chain and preserve the $U_q(SU_2)$ invariance. A construction of a Hamiltonian was proposed by [2]. Couplings of nearest-neighbour spins in a quantum group invariant way is easily done and yields the XXZ model. Through an invariant coupling of the last spin with the first one, a 'non-local' boundary term is added. The resulting Hamiltonian, which we treat in this paper, is given by

$$H := Lq - \sum_{i=1}^{L-1} R_i - R_0$$
$$R_0 := G R_{L-1} G^{-1} \qquad\qquad (1.1)$$
$$G := R_1 R_2 \ldots R_{L-1}.$$

It describes a spin chain of length L. To each lattice point a spin-$\frac{1}{2}$ degree of freedom is attached. R_i are 4×4 matrices giving a nearest-neighbour interaction

$$R_i = \sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ + \frac{\rho}{2}(\sigma_i^3 \sigma_{i+1}^3 + 1) - \frac{q - q^{-1}}{4}(\sigma_i^3 - \sigma_{i+1}^3 - 2) \quad (1.2)$$

where $\rho = (q + q^{-1})/2$, and q denotes the deformation parameter which becomes, at a later stage, a root of unity. R_is are a representation of the Hecke algebra

$$R_i^2 = (q - q^{-1})R_i + 1 \qquad R_i R_{i\pm1} R_i = R_{i\pm1} R_i R_{i\pm1} \qquad (1.3)$$

and therefore the quantum group invariance of (1.1) is evident. H commutes with the generators of $U_q(SU_2)$. Special degeneracies are connected to this symmetry. In addition, there exists a generator G, which is a substitute for the momentum. It shifts the R_is by one unit and maps R_0 into R_1:

$$G R_i G^{-1} = R_{i+1} \qquad R_L := R_0 \qquad i = 1, \ldots, L - 1. \qquad (1.4)$$

To obtain (1.4), the Hecke algebra conditions (1.3) have been used. A straightforward consequence of (1.4) is that $[G, H] = 0$. For $q = 1$, R_i becomes the permutation operator for lattice spins i and $i + 1$, $R_i = P_i$ and R_0 exchanges the Lth spin with the first. Equation (1.1) then becomes the usual $SU(2)$-invariant closed XXX spin chain.

Defining the unitary operator U by

$$U = \exp\left(i\frac{\pi}{2} \sum_{j=1}^{L} j\sigma_j^3\right) \qquad (1.5)$$

one has $U R_i(q) U^{-1} = -R_i(-q^{-1})$. The Hamiltonian (1.1) therefore has the property

$$U H(q) U^{-1} = -H(-q^{-1}). \qquad (1.6)$$

We intend to diagonalize (1.1) for $q = e^{i\varphi}$ being on the unit circle. In section 2 we show that the Bethe method works, although the Bethe equations have to be slightly modified. Special boundary conditions result, which depend on the Bethe spin. In section 3 we analyse the ground-state properties of this chain. As a function of φ or $\gamma = \pi - \varphi$ an interesting phenomenon shows up. For $0 < \gamma < \pi/2$ the ground state has spin zero. At $\gamma = \pi/2$ it becomes degenerate and for $\gamma > \pi/2$ the total spin of the ground state is non-zero and depends on γ. This property has been confirmed by diagonalizing small chains and by numerical methods and we conjecture it to hold for every finite chain. In section 4 we present the finite-size scaling behaviour which shows non-trivial dependence on γ. Finally, in section 5 we discuss some properties of related models.

2. Bethe states

We introduce a basis of the 2^L-dimensional Hilbert space $\mathcal{H} = \otimes \mathbb{C}^2$ in which the Hamiltonian (1.1) acts. Denote by $|0\rangle$ the ferromagnetic state with all spins pointing up and by $|n_1, \ldots, n_M\rangle$ the state where at the positions n_i a spin is pointing down, while all others point up. From equation (1.2) we have

$$R_i|0\rangle = q|0\rangle \qquad G|0\rangle = q^{L-1}|0\rangle \qquad H|0\rangle = 0. \qquad (2.1)$$

The Hamiltonian of our model commutes with the third component of the total spin operator J^3 and therefore the number operator counting down spins $M = L/2 - J^3$ is conserved. A general vector $|\psi_M\rangle \in \mathcal{H}$ with M spins pointing down can be expressed as

$$|\psi_M\rangle = \sum_{1 \leqslant n_1 < n_2 \cdots < n_M \leqslant L} \psi_M(n_1, \ldots, n_M)|n_1, \ldots, n_M\rangle. \qquad (2.2)$$

Equation (1.1) will be diagonalized using Bethe states for the wavefunctions in (2.2). We follow the standard strategy and determine first restrictions on ψ_M. Let Δ be the coproduct of $U_q(SU_2)$ which acts on the generators $J^\pm$, J^3 as

$$\Delta(J^\pm) = q^{-J^3} \otimes J^\pm + J^\pm \otimes q^{J^3} \qquad \Delta(q^{J^3}) = q^{J^3} \otimes q^{J^3}. \qquad (2.3)$$

With the help of Δ^L defined iteratively by $\Delta^{n+1} = (\Delta \otimes \mathrm{id}) \circ \Delta^n$ the representation for $U_q(SU_2)$ in $\mathcal{H}_L$ is given by

$$J^\pm = \Delta^L(\sigma^\pm) \qquad q^{J^3} = \Delta^L(q^{\sigma^3/2}) \,. \tag{2.4}$$

For $M = 1$, the total spin is $L/2$ or $L/2 - 1$. States with spin $L/2 - 1$ are the highest members of their spin multiplets and therefore satisfy

$$\Delta^L(\sigma^+)|\psi\rangle = 0 \,. \tag{2.5}$$

Equation (2.5) yields the sum rule

$$\sum_{n=1}^{L} q^{-n} \psi_1(n) = 0 \tag{2.6}$$

which has as solutions

$$\psi_1(n) = \mathrm{constant} \times q^n z^n \tag{2.7}$$

where z is one of the $L - 1$ non-trivial Lth roots of unity with $z^L = 1$, $z \neq 1$. Different roots distinguish different $U_q(SU_2)$ multiplets with spin $J = L/2 - 1$. If we extend the range of definition of the function ψ_1 to all integers, we obtain from (2.7) the boundary condition

$$\psi_1(n + L) = q^L \psi_1(n) \,. \tag{2.8}$$

We can equally well use quasimomenta k defined by $e^{ik} = qz$. The allowed values of k are $k_i = 2\pi i/L + \varphi$, where $i = 1, \ldots, L - 1$. $i = 0$ would correspond to $J^-|0\rangle$. In order to solve the eigenvalue equation for (1.1) next we need the action of R_i and G on the basis of $\mathcal{H}_L$:

$$R_i|j\rangle = q|j\rangle \quad j \neq i, i+1 \qquad R_i|i\rangle = |i+1\rangle + (q - q^{-1})|i\rangle \qquad R_i|i+1\rangle = |i\rangle \,. \tag{2.9}$$

A straightforward calculation using (2.6)–(2.8) allows us to deduce the eigenvalues of G and H on $|\psi_1\rangle$:

$$G|\psi_1\rangle = q^{L-3} z |\psi_1\rangle \qquad H|\psi_1\rangle = E|\psi_1\rangle \tag{2.10}$$

with $E_i = 2(\rho - \cos k_i)$. Let us remark that for $q = e^{i\varphi}$ the energy E_i of the 'one particle' state $|k_i\rangle$ fulfils

$$E_i > 0 \iff \varphi < \frac{\pi}{L} = \frac{\pi}{2s_{\max}} \,. \tag{2.11}$$

In conclusion, the ferromagnetic state $|0\rangle$ has lower energy then the one particle highest weight states for this values of φ!

The calculation for general M is more tedious. We start again with the $U_q(SU_2)$ sum rule. States satisfying $J^+|\psi_M\rangle = 0$ have spin $J = L/2 - M$ and $J^3 = J$. Application of J^+ yields

$$\sum_{n=1}^{n_2-1} \psi_M(n, n_2, \ldots, n_M) q^{-n-M+1}$$

$$+ \sum_{j=2}^{M-1} \sum_{n=n_{j-1}+1}^{n_j-1} \psi_M(n_2, n_3, \ldots, n_{j-1}, n, n_{j+1}, \ldots, n_M) q^{-n-M+2j-1}$$

$$+ \sum_{n=n_M+1}^{L} \psi_M(n_2, \ldots, n_M, n) q^{-n+M-1} = 0 \,. \tag{2.12}$$

We shall try to satisfy this sum rule with the help of the ansatz

$$\psi_M(n_1, \ldots, n_M) = \sum_{P \in S_M} A_P \, t_{P_1}^{n_1 - 1} \ldots t_{P_M}^{n_M - 1} \tag{2.13}$$

where the sum runs over all elements P of the permutation group S_M. Quasimomenta k_i are defined as before through $t = e^{ik}$. Under the conditions that $t_i \neq q$ we obtain

$$A_P = q^{2M - L - 2} \, t_{P_1}^L A_{PC} \qquad \text{and} \qquad A_{P'} = -A_P \, \frac{2\rho t_{P_{i+1}} - t_{P_i} t_{P_{i+1}} - 1}{2\rho t_{P_i} - t_{P_i} t_{P_{i+1}} - 1} \tag{2.14}$$

where C denotes the cyclic permutation $C(1, 2, \ldots, M) = (2, \ldots, M, 1)$ and P' denotes the transposition of neighbours in P: $P' = (P_1, P_2, \ldots, P_{i+1}, P_i, \ldots, P_M)$. Applying the first equation of the (2.14) M times, we obtain

$$(t_{P_1} \ldots t_{P_M})^L = q^{M(L - 2M + 2)}. \tag{2.15}$$

From combining (2.15) and (2.13) we obtain the boundary condition for ψ_M:

$$\psi_M(n_2, \ldots, n_M, n_1 + L) = q^{L + 2 - 2M} \psi_M(n_1, \ldots, n_M). \tag{2.16}$$

Next we study the action of the operator G and show that it is diagonal on Bethe states, $G|\psi_M\rangle = \Gamma|\psi_M\rangle$. G applied to $|\psi_M\rangle$ produces a state with components

$$\begin{aligned} (G\psi_M)(1, n_2 + 1, \ldots, n_M + 1) &= q^{M-3} \psi_M(n_2, \ldots, n_M, L) \\ (G\psi_M)(n_1 + 1, \ldots, n_M + 1) &= q^{L-1-M} \psi_M(n_1, \ldots, n_M). \end{aligned} \tag{2.17}$$

The eigenvalue problem for G can now be solved by combining (2.13) and conditions (2.14) and (2.15). The corresponding eigenvalue Γ satisfies

$$\Gamma^L = q^{L(L-1) - 2M(L - M + 1)}. \tag{2.18}$$

Now we diagonalize the Hamiltonian. The energy eigenvalue equations can be written as

$$(2M\rho - E)\psi_M(n_1, \ldots, n_M) - \sum_{j=\pm 1} \sum_{k=1, n_k \neq n_{k-1} + j}^{M} \psi_M(n_1, \ldots, n_k - j, \ldots, n_M) = 0 \tag{2.19}$$

where unwanted terms have not been included. They cancel if

$$\psi_M(n_k + 1, n_k + 1, \ldots) + \psi_M(n_k, n_k) - 2\rho \psi_M(n_k, n_k + 1) = 0. \tag{2.20}$$

Equations (2.19) and (2.20) have to be combined with the boundary condition (2.16). Inserting the ansatz (2.13) into (2.19) yields for the energy eigenvalues

$$E = 2 \sum_{i=1}^{M} (\rho - \cos k_i) \tag{2.21}$$

where the k_is have to be determined by the Bethe equations following from (2.20). Equations (2.13)–(2.16), (2.19) and (2.20) yield eigenfunctions for spin $J = L/2 - M$ and $J^3 = L/2 - M$ to eigenvalue (2.21). They are also eigenfunctions of the operator G with eigenvalues given by (2.18).

We define in addition scattering phase shifts $\Theta(k_i, k_j)$ by

$$\begin{aligned} e^{i\Theta(k_i, k_j)} &= -\frac{e^{i(k_i + k_j)} + 1 - 2\rho \, e^{ik_i}}{e^{i(k_i + k_j)} + 1 - 2\rho \, e^{ik_j}} \\ \Theta(k_i, k_j) &= 2 \arctan \left[\frac{\rho \sin(k_i - k_j)/2}{\cos(k_i + k_j)/2 - \rho \cos(k_i - k_j)/2} \right]. \end{aligned} \tag{2.22}$$

In this notation (2.14) becomes

$$\frac{A_P}{A_{PC}} = q^{2M-L-2} e^{ik_1 L} \tag{2.23}$$

and

$$A_{P'} = -A_P \, e^{-i\Theta(k_{P_{i+1}}, k_{P_i})} . \tag{2.24}$$

This equation can be iterated further. Taking into account that (2.23) and (2.24) hold for all permutations, we obtain the Bethe equations for k_i which are in their logarithmic form given by

$$Lk_i + \varphi(2M - L - 2) + \sum_{j=1}^{M} \Theta(k_i, k_j) = 2\pi I_i . \tag{2.25}$$

Where I_i are integers (half integers) if M is odd (even).

3. Ground-state properties

We start from (2.25) and relabel φ in terms of $\gamma = \pi - \varphi, 0 \leqslant \gamma \leqslant \pi$, $I_i^{\gamma} = I_i + L/2 - M + 1$. Then

$$Lk_i = 2\pi I_i^{\gamma} + \gamma(2M - L - 2) - \sum_{j=1}^{M} \Theta(k_i, k_j) . \tag{3.1}$$

For $\gamma = 0$ our Hamiltonian coincides with the usual XXX Hamiltonian. The Bethe constraints (3.1) coincide too. We expect therefore that the standard choice for the Bethe numbers:

$$I_i^{\gamma} = -\left(\frac{M-1}{2}\right), \dots, \left(\frac{M-1}{2}\right) \tag{3.2}$$

or

$$I_i = -\left(\frac{M-1}{2}\right) - 1, \dots, \left(\frac{M-1}{2}\right) - 1 \tag{3.3}$$

will still give the ground state for an interval in γ. We confirmed that the non-degenerate ground state is obtained in this way for $0 \leqslant \gamma < \pi/2$ in a number of cases. Numerical diagonalization was done for small chains with $L \leqslant 10$ and confirms that this is the non-degenerate ground state. A difference to the periodic XXZ chain shows up for $\gamma \geqslant \pi/2$.

We have a simple argument that already at $\gamma = \pi/2$ a degeneracy occurs: $\Theta(k_i, k_j)$ vanishes for this point and the Bethe equations become $Lk_i = 2\pi(I_i^{\gamma} - \frac{1}{2})$. In order to minimize the energy $E = -2\sum_i \cos k_i$, we choose the k_i as dense as possible around $k = 0$. Since there are two zero-modes at $k = \pm\pi/2$ two possibilities occur: either

$$k_i = -\frac{\pi}{2}, \dots, \frac{\pi}{2} - \frac{2\pi}{L} \qquad \text{for} \quad M = L/2 \tag{3.4}$$

or

$$k_i = -\frac{\pi}{2} + \frac{2\pi}{L}, \dots, \frac{\pi}{2} - \frac{2\pi}{L} \qquad \text{for} \quad M = L/2 - 1 . \tag{3.5}$$

Both choices lead to the same energy. Therefore we obtain a degeneracy due to states at the band edge. Note that (3.4) and (3.5) can be obtained by choosing I_i according to (3.3).

An exact diagonalization for $N = 4$ gives levels drawn in figure 1. The same has been done for the Pasquier–Saleur Hamiltonian [1]. In both cases $J^3 = 1$ results as the spin of
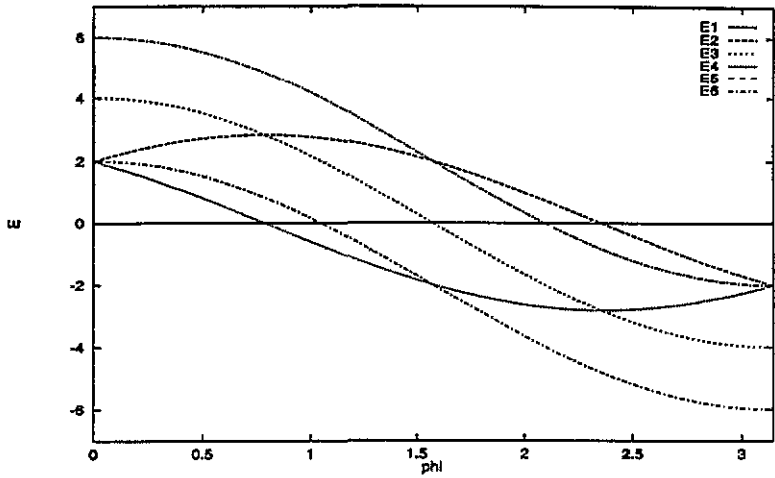
Figure 1. Dependence of the energy levels of the model (1.1) for $L = 4$, $M = 2$. E_1 belongs to the $U_q(SU(2))$ quintuplet. E_2, E_3, E_4 belong to the triplets. E_5 and E_6 are singlets. The ground-state changes from the s sector into spin s at $\varphi = \pi/2s$.

the ground state if $\pi/2 < \gamma < \pi/4$. This is in striking contrast to the XXZ chain with periodic boundary conditions. In this case a unique ground state is obtained for all values of γ as it must be for all finite L according to a theorem by Affleck and Lieb [3].

A detailed analysis led us to the following.

Conjecture. For any finite L (even), the total spin J of the ground state depends on the value of the anisotropy φ according to

$$
\begin{aligned}
J &= 0 && \text{for} \quad \frac{\pi}{2} < \varphi < \pi \\
J &= s && \text{for} \quad \frac{\pi}{2(s+1)} < \varphi < \frac{\pi}{2s} \\
J &= \frac{L}{2} && \text{for} \quad 0 < \varphi < \frac{\pi}{L}.
\end{aligned}
\tag{3.6}
$$

The ground state is non-degenerate (up to the trivial $U_q(SU_2)$ degeneracy). At the edges of the intervals, $\varphi = \pi/2s$, additional degeneracies occur.

In terms of Bethe numbers, the ground state belongs to the set (3.3) with $M = L/2 - s$ chosen according to (3.6).

Evidence for the conjecture is given by diagonalizing small chains and solving the Bethe equations numerically for the special set of numbers I_i given in (3.3). In addition the finite-size analysis presented in the next chapter agrees with the conjecture.

Let us now analyse the behaviour of our model in the thermodynamic limit $L \to \infty$. We shall essentially follow [4]. Changing variables from k to λ and defining

$$
\Phi(\lambda, \alpha) = 2 \tan^{-1}[\cot \alpha \tanh \lambda]
\tag{3.7}
$$

we have

$$
k_i = \Phi(\lambda_i, \gamma/2) \qquad \Theta(k_i, k_j) = -\Phi(\lambda_i - \lambda_j, \gamma).
\tag{3.8}
$$

The Bethe constraints (3.1) now read

$$\frac{1}{2\pi}\left[\Phi(\lambda_i,\gamma/2)-\frac{1}{L}\sum_{i=1}^{M}\Phi(\lambda_i-\lambda_j,\gamma)+\frac{2\gamma}{L}(s+1)\right]=\frac{J_i}{L}. \tag{3.9}$$

As usual, we define a function $Z(\lambda,\gamma)$ by the left-hand side of (3.9):

$$Z(\lambda,\gamma)=\frac{1}{2\pi}\left[\Phi(\lambda,\gamma/2)-\frac{1}{L}\sum_{j=1}^{M}\Phi(\lambda-\lambda_j,\gamma)+\frac{2\gamma}{L}(s+1)\right]. \tag{3.10}$$

Introducing the density of roots $\sigma_L(\lambda):=\mathrm{d}Z_N(\lambda,\gamma)/\mathrm{d}\lambda$ which is independent of the shift $2\gamma(\Sigma+1)$, we obtain the usual integral equation for σ_∞ of the XXZ model with solution (see [4] for the details)

$$\sigma_\infty(\lambda)=\frac{1}{2\gamma\cosh(\pi\lambda/\gamma)}. \tag{3.11}$$

The energy per spin $e_\infty=\lim E_L/L$ is given by the expression

$$e_\infty=-\sin^2\gamma\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda}{\cosh\pi\lambda}\frac{1}{\cosh 2\gamma\lambda-\cos\gamma}. \tag{3.12}$$

4. Finite-size corrections

It may be expected, that the peculiar features of the ground state will be reflected in the finite-size scaling behaviour of the model. Again, we follow the derivation of [4].

The deviation of the finite chain from the thermodynamic limit can be described by

$$\sigma_L(\lambda)-\sigma_\infty(\lambda)=-\int_{-\infty}^{\infty}\frac{\mathrm{d}\mu}{\pi}p(\lambda-\mu)\left(\frac{1}{L}\sum_{i=1}^{M}\delta(\mu-\lambda_i)-\sigma_L(\mu)\right) \tag{4.1}$$

$$e_L-e_\infty=-2\pi\sin\gamma\int_{-\infty}^{\infty}\mathrm{d}\lambda\sigma_\infty(\lambda)\left(\frac{1}{L}\sum_{i=1}^{M}\delta(\lambda-\lambda_i)-\sigma_L(\lambda)\right). \tag{4.2}$$

The kernel $p(\lambda)$ is given by

$$\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\mathrm{e}^{\mathrm{i}\lambda\omega}\frac{\sinh(\pi-2\gamma)\omega/2}{\sinh(\pi-2\gamma)\omega/2+\sinh(\pi\omega/2)}. \tag{4.3}$$

Approximating the integrals in (4.1) and (4.2) by the Euler–Maclaurin formula, the system of equations relevant for the finite-size behaviour gives

$$\int_{\Lambda^+}^{\infty}\sigma_L(\lambda)\mathrm{d}\lambda=Z_L(\infty)-Z_L(\Lambda^+) \tag{4.4}$$

$$\int_{-\infty}^{-\Lambda^-}\sigma_L(\lambda)\mathrm{d}\lambda=Z_L(-\Lambda^-)-Z_L(-\infty) \tag{4.5}$$

$$\sigma_L(\lambda)-\sigma_\infty(\lambda)=\int_{\Lambda^+}^{\infty}\mathrm{d}\mu\sigma_L(\mu)\frac{p(\lambda-\mu)}{\pi}+\int_{-\infty}^{-\Lambda^-}\mathrm{d}\mu\sigma_L(\mu)\frac{p(\lambda-\mu)}{\pi}$$
$$-\frac{1}{2\pi L}p(\lambda-\Lambda^+)+\frac{1}{12\pi L^2\sigma_L(\Lambda^+)}p'(\lambda-\Lambda^+)$$
$$-\frac{1}{2\pi L}p(\lambda+\Lambda^-)+\frac{1}{12\pi L^2\sigma_L(-\Lambda^-)}p'(\lambda+\Lambda^-) \tag{4.6}$$
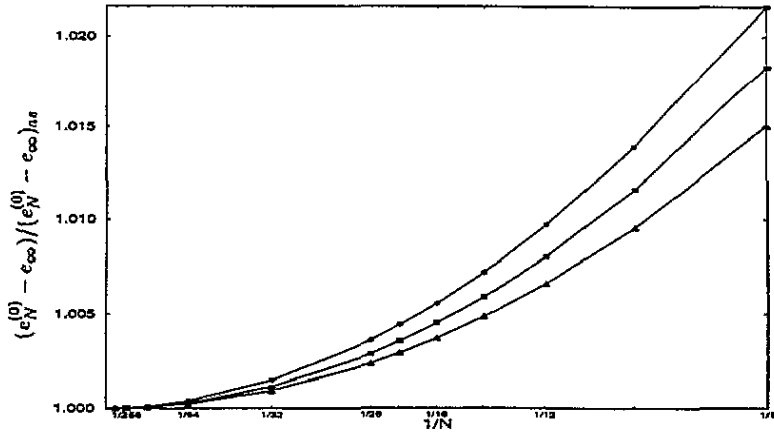
Figure 2. $e_L^0 - e_\infty$ describes the difference between the energy per site of the finite chain and the thermodynamic limit calculated numerically from the Bethe equations. $(e_L^0 - e_\infty)_{as}$ describes the same quantity as predicted from the analytical formula (4.11). The ratio is plotted for three different values of φ. We see that the agreement between these two approaches increases with the length of the chain.

$$e_L - e_\infty = 2\pi \sin \gamma \left[\int_{\Lambda^+}^{\infty} d\lambda \sigma_\infty(\lambda) \sigma_L(\lambda) - \frac{\sigma_\infty(\Lambda^+)}{2L} - \frac{\sigma_\infty'(\Lambda^+)}{12L^2 \sigma_L(\Lambda^+)} \right.$$
$$\left. + \int_{-\infty}^{-\Lambda^-} d\lambda \sigma_\infty(\lambda) \sigma_L(\lambda) - \frac{\sigma_\infty(-\Lambda^-)}{2L} - \frac{\sigma_\infty'(-\Lambda^-)}{12L^2 \sigma_L(-\Lambda^-)} \right] \tag{4.7}$$

where Λ^+ (Λ^-) denote the maximal (minimal) root λ of the Bethe equations given in (3.9). From the definition of $Z_L(\lambda, \gamma)$ we obtain

$$Z_L(\infty) = \frac{1}{4} - \frac{\varphi}{\pi L} - \frac{s}{2L}$$
$$Z_L(-\infty) = -\frac{1}{4} - \frac{\varphi}{\pi L} + \frac{s}{2L} - \frac{2\varphi s}{\pi L}$$
$$Z_L(\Lambda^+) = \frac{1}{4} - \frac{s}{2L} - \frac{3}{2L}$$
$$Z_L(-\Lambda^-) = -\frac{1}{4} + \frac{s}{2L} - \frac{1}{2L} \tag{4.8}$$

and

$$\int_{\Lambda^+}^{\infty} \sigma_L(\lambda) \, d\lambda = \frac{1}{2L} \left(1 - \frac{\psi_+}{\pi} \right)$$
$$\int_{-\infty}^{-\Lambda^-} \sigma_L(\lambda) \, d\lambda = \frac{1}{2L} \left(1 + \frac{\psi_-}{\pi} \right). \tag{4.9}$$

Using equation (3.3), we obtain for $\psi_\pm$

$$\psi_+ = -2\gamma \qquad \psi_- = -2\gamma(1 + 2s) + 4s. \tag{4.10}$$

From equations (4.6), (4.7), (4.9) and (4.10) the finite-size correction to the energy can be

computed and is given by

$$
e_L - e_\infty = -\frac{\pi^2}{6L^2}\frac{\sin\gamma}{\gamma}\left[1 - \frac{3}{4\pi(\pi-\gamma)}(\psi_+^2 + \psi_-^2)\right]
$$

$$
= -\frac{\pi^2}{6L^2}\frac{\sin\gamma}{\gamma}\left[1 - \frac{6}{\pi(\pi-\gamma)}[\gamma^2 - 2\gamma s(\pi-\gamma) + 2s^2(\pi-\gamma)^2]\right]. \qquad (4.11)
$$

Remember that γ and s are linked by (3.6). Defining c by $e_L - e_\infty = -(\pi^2/6L^2)(\sin\gamma/\gamma)c$ we see that the conformal anomaly of the model (1.1) can be expressed by

$$
c = 1 - \frac{6}{\pi(\pi-\gamma)}\left[\gamma^2 - 2\gamma s(\pi-\gamma) + 2s^2(\pi-\gamma)^2\right]
$$

$$
= 1 - \frac{6}{\pi\varphi}\left[(\pi-\varphi)^2 - 2s\varphi(\pi-\varphi) + 2s^2\varphi^2\right]. \qquad (4.12)
$$

For $0 < \gamma < \pi/2$, $(s=0)$, we recover the usual conformal anomaly

$$
c = 1 - \frac{6\gamma^2}{\pi(\pi-\gamma)}. \qquad (4.13)
$$

In particular, for $\gamma = \pi/m$ with $m = 3, 4, \ldots$, c belongs to the unitary series $c = 1 - 6/m(m-1)$. We expect that the Hamiltonian (1.1) gives (after rescaling by a factor $\pi^2 \sin\gamma/\gamma$) a conformal field theory in the thermodynamic limit.

Note that in the region of γ where non-zero spin states appear as ground states, we have $c < 0$ and therefore the model will not correspond to a unitary representation of the Virasoro algebra.

Finally let us make some remarks about the expression (4.12). The energy difference of states with spin s and spin $s+1$ follows from (4.12) and is given by

$$
e_L^s - e_L^{s+1} = (e_L^s - e_\infty) - (e_L^{s+1} - e_\infty)
$$

$$
= \frac{2\pi}{L^2}\sin\gamma\left[1 - \frac{(2s+1)\varphi}{\pi-\varphi}\right]. \qquad (4.14)
$$

This function has zeros precisely at the boundaries of the intervals $\varphi = \pi/(2s+1)$ given in the conjecture (3.6). Both states become degenerate at these points. The state with spin $s+1$ becomes more stable ($e_L^s - e_L^{s+1} > 0$) for $\varphi < \pi/2s$ as predicted by our conjecture.

5. Relations to other models

As we said at the beginning of this paper our aim was to study a quantum group invariant model which is also periodic on a lattice with L sites. It is therefore interesting, but not surprising, to remark that this model can be obtained in various other ways. It was obtained in [5] following a Hamiltonian approach to quantize the Chern–Simons model on a cylinder. The model was also discussed in [6], although their notation was slightly different. In [6] the algebraic nested Bethe ansatz has been worked out. We would also like to mention, that the form of the last contribution to our Hamiltonian R_0 is dictated by requiring covariance under quantum group transformations [7].

It is known [9, 11], that the XXZ model with toroidal boundary conditions is related to various other models of statistical physics with $c < 1$. The natural question is, what the corresponding models are in our case. In fact, the mapping can be based on the observation [10, 12], that the Hamiltonians can be expressed in terms of different representations of the Temperley–Lieb algebra. This is the case, for example, for the periodic XXZ chain of $2L$ sites for $q = e^{i\pi/4}$ and the Ising model for L sites or for the periodic XXZ chain

at $q = e^{i\pi/6}$ and the L-site 3-state Potts model. We shall follow the same idea. We rewrite our Hamiltonian in terms of Temperley–Lieb generators. Then we shall explore the same expression but in the representations which are known to be appropriate for the Ising or Potts model. To be more specific, we start from the already mentioned fact, that the Hamiltonian of our model can be written in terms of Temperley–Lieb generators $e_i = q - R_i$, $i = 1, \ldots, L - 1$. The representation is defined with (1.2). The non-local boundary term becomes

$$q - R_0 = q - G(q - e_{L-1})G^{-1}$$
$$G = (q - e_1) \ldots (q - e_{L-1}). \tag{5.1}$$

We shall choose $q = e^{i\pi/4}$ or $m = 3$, $c = \frac{1}{2}$ which fits with the Ising case and take a well known representation of e_is:

$$e_{2j} = \frac{1}{\sqrt{2}}(1 + \sigma_i^x \sigma_{i+1}^x) \qquad e_{2j-1} = \frac{1}{\sqrt{2}}(1 + \sigma_i^3). \tag{5.2}$$

After a straightforward calculation we obtain the Hamiltonian

$$H = -\sum_{i=1}^{L-1}(e_{2j} + e_{2j-1}) - e_{2L}$$
$$= -\frac{1}{\sqrt{2}}\left[\sum_{i=1}^{L-1}\sigma_i^x \sigma_{i+1}^x + \sum_{i=1}^{L}\sigma_i^3\right] - e_{2L} - \frac{2L-1}{\sqrt{2}}. \tag{5.3}$$

where the 'boundary term' e_{2L} calculated again according to (5.1) becomes

$$e_{2L} = \frac{1}{\sqrt{2}}(1 + \sigma_L^x \Gamma \sigma_1^x) \qquad \Gamma := -\sigma_1^3 \ldots \sigma_L^3. \tag{5.4}$$

This means, that according to the two possible eigenvalues $(-1)^\gamma$, $\gamma = 0, 1$ of Γ we deal either with periodic or antiperiodic boundary conditions:

$$\sigma_{L+1}^x = (-1)^\gamma \sigma_1^x. \tag{5.5}$$

Within this model definite boundary conditions are prescribed for a particular sector. We expect that the two models have related spectra. Indeed we have checked numerically for $L = 2, 3$ (the first case of course can easily be solved analytically) that the Ising spectrum is contained in the spectrum of the XXZ Hamiltonian.

We are quite familiar with a string-like Γ from the Jordan–Wigner transformation. Applying it to (5.4) yields a fermion Hamiltonian

$$H = -\frac{1}{\sqrt{2}}\left[\sum_{i=1}^{L}c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}^\dagger + c_{j+1}c_j + 2c_j^\dagger c_{j+1} + 1\right] \tag{5.6}$$

with antiperiodic boundary conditions $c_{L+1} = -c_1$ for L even.

We may illustrate these remarks next for $m = 5$ or $q = e^{i\pi/6}$. The 3-state Potts model fits to the conformal charge $c = \frac{4}{5}$. The appropiate representation of the Temperley–Lieb algebra becomes

$$e_{2j} = \frac{1}{\sqrt{3}}(1 + \Gamma_j \Gamma_{j+1}^\dagger + \Gamma_j^\dagger \Gamma_{j+1}) \qquad e_{2j-1} = \frac{1}{\sqrt{3}}(1 + \sigma_j + \sigma_j^\dagger)$$

$$\sigma_j = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.7}$$

with $\sigma\Gamma = \omega\Gamma\sigma$, $\sigma^3 = \Gamma^3 = 1$. The resulting Hamiltonain is

$$H_0 = -\sum_{j=1}^{2N-1} e_j = -\frac{1}{\sqrt{3}}\sum_{j=1}^{N-1}(\Gamma_j\Gamma_{j+1}^\dagger + \Gamma_j^\dagger\Gamma_{j+1}) + \sum_{j=1}^{N}(\sigma_j + \sigma_j^\dagger) - \frac{2N-1}{\sqrt{3}}. \tag{5.8}$$

The boundary term $H' = -e_{2N}$, which is added to H_0 yields the total Hamiltonian $H = H_0 + H'$, given explicitly as

$$H' = \frac{1}{\sqrt{3}}\left(1 + M\Gamma_1\Gamma_N^\dagger + \Gamma_1^\dagger\Gamma_N M^\dagger\right) \tag{5.9}$$

where $M = \sigma_1 \ldots \sigma_N$. The operator M has eigenvalues $1, \omega$ and ω^2 and therefore we again obtain a model with boundary conditions

$$\Gamma_{N+1} = \omega^\gamma\Gamma_1 \qquad \gamma = 0, 1, 2 \tag{5.10}$$

depending on the sector corresponding to the eigenvalues of M. As in the previous case, we have checked for $L = 2, 3$ that the Potts model spectrum is contained in the spectrum of the XXZ chain.

The relation between the twisted $2L$-sites XXZ model and the 3-states periodic Potts model for L sites was found in [10] but only for the ground-state sector. Now we have the generalization of the previous statement without the limitation to the ground-state sector.

In summarizing, we diagonalized a one parameter family of Hamiltonians of a quantum group invariant and periodic model. We obtained a sequence of ground states with increasing spin. These transitions resemble the incommensurate transition obtained in various other models [13].

Acknowledgments

References

[1] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523
[2] Martin P P 1991 *Potts Models and Related Problems in Statistical Mechanics* (Singapore: World Scientific)
[3] Affleck I and Lieb E 1986 *Lett. Math. Phys.* **12** 57
[4] Hamer C J, Quispel G R W and Batchelor M T 1987 *J. Phys. A: Math. Gen.* **20** 5677
[5] Alekseev A Yu 1994 Integrability in the Hamiltonian Chern–Simons theory *Uppsala preprint*
[6] Karowski M and Zapletal A 1994 Quantum group invariant integrable n-state vertex models with periodic boundary conditions *Preprint* SFB 288 Nr 97
[7] Scheunert M 1994 The tensor product of tensor operators over quantum algebras: some applications to quantum spin chains *Preprint* Bonn-94-0042
[8] Lieb E, Schultz T and Mattis D 1961 *Ann. Phys.* **16** 407
[9] Alcaraz F C, Grimm U and Rittenberg V 1989 *Nucl. Phys.* B **316** 735
[10] Alcaraz F C, Barber M N and Batchelor M T 1989 *Ann. Phys.* **182** 280
[11] Grimm U and Schütz G 1992 The spin-$\frac{1}{2}$ XXZ chain, the quantum algebra $U_q[sl(2)]$ and duality transformations for minimal models *Preprint* WIS-92/75/Sept-PH
[12] Hinrichsen H, Martin P P, Rittenberg V and Scheunert M 1993 On the two point correlation functions for the $U_q[SU(2)]$ invariant spin one-half Heisenberg chain at roots of unity *Preprint* Bonn HE–93–35
[13] Hoeger C, Gehlen G v and Rittenberg V 1985 *J. Phys. A: Math. Gen.* **18** 1813